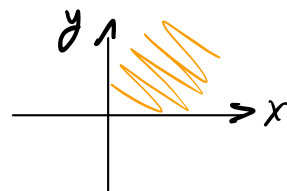


Due Fri

4.5 - Coordinates and Basis



Definition: If $S = \{v_1, v_2, \dots, v_n\}$ is a set of vectors in a finite-dimensional vector space V , then S is called a **basis** for V if:

a) S spans V . 4.3

b) S is linearly independent. 4.4



$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are standard basis vectors for \mathbb{R}^3

These relate to mutually perpendicular axes.

However, any 3 vectors in \mathbb{R}^3 that do not lie on the same plane can be used to represent any other vector in \mathbb{R}^3 . These 3 vectors are called basis vectors.

#1 Use the determinant of a coefficient matrix to show that the following set of vectors forms a basis for \mathbb{R}^2 : $\{(2, 1), (3, 0)\}$.

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$. If $A\vec{x} = \vec{b}$ always has a solution, then the set spans \mathbb{R}^2 . If $A\vec{x} = \vec{0}$ has only the trivial solution, then the set is independent. By Equivalent Statements, this is shown if $\det(A) \neq 0$.

Here, $\det(A) = -3 \neq 0$. The set is a basis for \mathbb{R}^2 .

#8 Show that the following vectors do not form a basis for P_2 .

$1 - 3x + 2x^2, 1 + x + 4x^2, 1 - 7x$

$$\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 1(2 - 28) - (-26) = 0$$

$0 - 14 - 12$

So the vectors do not form a basis for P_2 .

Note: $2(1 - 3x + 2x^2) - (1 + x + 4x^2) = 1 - 7x$

Definition: A basis in which the listed order of the vectors matters is called an **ordered basis**.

Some Standard bases

$\{\hat{i}, \hat{j}\}$ is a basis for R^2 (this is the same as $\{e_1, e_2\}$).

$\{\hat{i}, \hat{j}, \hat{k}\}$ is a basis for R^3 (this is the same as $\{e_1, e_2, e_3\}$).

$\{e_1, e_2, \dots, e_n\}$ is a basis for R^n .

$\{1, x, x^2, \dots, x^n\}$ is a basis for P_n .

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for M_{22} .

Space	Basis
R^2	$\{\vec{e}_1, \vec{e}_2\}$
R^3	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
P_2	$\{1, x, x^2\}$
$\{\vec{0}\}$	\emptyset

The basis for the zero vector space $V = \{\vec{0}\}$ is \emptyset (the empty set).

Theorem 4.5.1 Uniqueness of Basis Representation \Rightarrow

If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V , then every vector v in V can be expressed in the form $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ in exactly one way.

Spans $\hat{\wedge}$ lin. indep.

PF: let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for

a vector space V . Let $\vec{v} \in V$. Since S is a basis, S spans V , so $\exists c_1, c_2, \dots, c_n \exists$

$$\vec{v} = \sum_{i=1}^n c_i \vec{v}_i. \text{ Suppose } \exists k_1, k_2, \dots, k_n \exists$$

$$\vec{v} = \sum_{i=1}^n k_i \vec{v}_i. \quad \vec{v} - \vec{v} = \sum_{i=1}^n c_i \vec{v}_i - \sum_{i=1}^n k_i \vec{v}_i$$

$$\vec{v} + (-\vec{v}) = \vec{0}$$

$$\Rightarrow \vec{0} = \sum_{i=1}^n (c_i - k_i) \vec{v}_i. \text{ Since } S \text{ is lin. indep.,}$$

$c_i - k_i = 0 \quad \forall i \Rightarrow c_i = k_i \quad \forall i$. So there is exactly one representation.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an ordered basis for a vector space V , and $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n$ is the expression for a vector \mathbf{v} in terms of the basis S , then the scalars c_1, c_2, \dots, c_n are called the **coordinates of \mathbf{v} relative to the basis S** . The vector (c_1, c_2, \dots, c_n) in R^n constructed from these coordinates is called the **coordinate vector of \mathbf{v} relative to the basis S** ; it is denoted by $(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$.

Example: Consider $\mathbf{v} = (-1, 7, 2) \in R^3$. The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2, -1, -1), \mathbf{v}_2 = (-2, 1, 2)$, and $\mathbf{v}_3 = (3, 5, 4)$ forms a basis for R^3 (verify). Find $(\mathbf{v})_S$ (where S is the standard basis for R^3) and $(\mathbf{v})_B$.

For $(\vec{v})_B$, we want $c_1, c_2, c_3 \exists c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{v}$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & -1 \\ -1 & 1 & 5 & 7 \\ -1 & 2 & 4 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

so $-6\vec{v}_1 - 4\vec{v}_2 + 1\vec{v}_3 = \vec{v}$. Thus

$$\boxed{(\vec{v})_B = (-6, -4, 1)}$$

using vectors in B

$$\boxed{(\vec{v})_S = (-1, 7, 2)} \text{ using vectors in } S$$

$$-6(2, -1, -1) - 4(-2, 1, 2) + 1(3, 5, 4)$$

$$\text{is the same as } -1(1, 0, 0) + 7(0, 1, 0) + 2(0, 0, 1)$$

#16 First show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for $M_{2,2}$, then express A as a linear combination of the vectors in S , and then find the coordinate vector of A relative to S .

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

Basis: $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

(or find determinant is not zero)

For $c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = A$,

$$c_2 = 2, c_3 = 3, c_1 + c_2 + c_3 = 6 \Rightarrow c_1 = 1$$

$$c_4 = 4 \quad A = 1A_1 + 2A_2 + 3A_3 + 4A_4$$

$$\text{so } (A)_S = (1, 2, 3, 4)$$

From 4.2 : Show that the following set is a subspace of \mathbb{R}^2 .

$$W = \{(x, y) \mid 2x + 3y = 0\}.$$

Theorem 4.2.1 Subspace Test

If W is a nonempty set of vectors in a vector space V , then W is a subspace of V if and only if the following conditions are satisfied.

- If \mathbf{u} and \mathbf{v} are vectors in W , then $\mathbf{u} + \mathbf{v}$ is in W .
- If k is a scalar and \mathbf{u} is a vector in W , then $k\mathbf{u}$ is in W .

Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2) \in W$.

(That is, $2u_1 + 3u_2 = 0$ & $2v_1 + 3v_2 = 0$.)

Then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$. So

$$\begin{aligned} 2(u_1 + v_1) + 3(u_2 + v_2) &= 2u_1 + 2v_1 + 3u_2 + 3v_2 \\ &= \underline{2u_1 + 3u_2} + \underline{2v_1 + 3v_2} \\ &= 0 + 0 = 0 \end{aligned}$$

So $\vec{u} + \vec{v} \in W$.

Let $k \in \mathbb{R}$. Then $k\vec{u} = (ku_1, ku_2)$ and

Let k be any scalar $2ku_1 + 3ku_2 = k(2u_1 + 3u_2) = k(0) = 0$

So $k\vec{u} \in W$. So W is a subspace.